quanticol

www.quanticol.eu

Population Modelling

Vashti Galpin
Laboratory for Foundations of Computer Science
University of Edinburgh

Open Problems in Concurrency Theory
Bertinoro

21 June 2014

COOPERATION

Modelling collective adaptive systems quantitatively

Motivation

- application area: collective adaptive systems (CAS)
  - smart transport – buses, bike sharing
  - smart grid – electricty generation and consumption
- we want model to quantitative behaviour of these systems and be able to characterise their performance
- we take a population-based approach where there are a large number of identical processes
- many processes leads to well-known problem of state space explosion
- mitigate this problem with approximation techniques
- focus in this talk on a general process algebra approach to modelling populations, moving beyond application to biology

Quantitative modelling

Modelling Language	$\mathbf{P}$
semantic mapping	a ↓
Mathematical Representation	$\mathbf{S}_a$
analysis technique	a_j ⇓
Result	$\mathbf{R}_{a_j}$

Modelling with PEPA

- PEPA [Hillston, 1996]
- two-level grammar, constant definition, $C \stackrel{def}{=} S$

$$S \quad ::= \quad (a, r).S \mid S + S$$
$$P \quad ::= \quad S \mid P \underset{L}{\bowtie} P$$

  multi-way synchronisation (CSP-style)
- operational semantics define labelled multi-transition system
  - $P_1 \xrightarrow{(a,r)} P_2$
  - labelled continuous-time Markov chain (CTMC)
- what happens when there are many sequential processes?
  - assume n sequential constants: $C_1, \ldots, C_n$
  - each constant has a maximum of m states: $S_{1,1}, \ldots, S_{1,m}$
  - CTMC has a maximum of n^m states

Modelling with PEPA

Modelling Language	**P**	$C_1 \bowtie_L \ldots \bowtie_L C_n$
semantic mapping	*CTMC* ↓	
Mathematical Representation	$\mathbf{S}_{CTMC}$	$(S_{1,j_1}, \ldots, S_{n,j_n})$
analysis technique	*steady state* ⇓	
Result	$\mathbf{R}_{CTMC}$	$(p_1, \ldots\ldots, p_{n^m})$

Quotienting by bisimilarity

- what if many of the sequential processes are the same?
- consider the states
  - $(S_{1,1}, S_{1,1}, S_{1,2}, S_{4,j_4}, \ldots, S_{n,j_n})$
  - $(S_{1,2}, S_{1,1}, S_{1,1}, S_{4,j_4}, \ldots, S_{n,j_n})$
  - both have the same numbers of $S_{1,1}$ and $S_{1,2}$
- numeric vector representation
  - $(\#S_{1,1}, \#S_{1,2} \ldots \#S_{1,m}; \ldots\ldots; \#S_{n',1}, \#S_{n',2}, \ldots \#S_{n',m})$
  - n' is number of different types of sequential constants
- introduces functional rates
- stochastically bisimilar
- smaller state space?
  - p is the maximum count of any state $S_{i,j}$
  - CTMC has a maximum of $(n' \times m)^{p+1}$ states

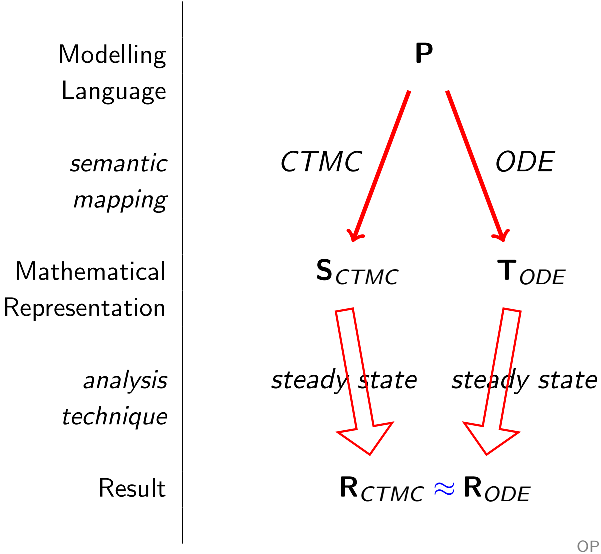
Using numeric vector representation

Modelling Language		$\mathbf{P}$	$C_1[x_1] \underset{L}{\bowtie} \ldots \underset{L}{\bowtie} C_{n'}[x_{n'}]$
semantic mapping	*CTMC*	$\downarrow$	
Mathematical Representation		$\mathbf{S}_{CTMC}$	$(\#S_{1,1}, \ldots; \ldots \#S_{n',m})$
analysis technique	*steady state*	$\Downarrow$	
Result		$\mathbf{R}_{CTMC}$	$(\, p_1 \,, \ldots\ldots, \; p_{(n' \times m)^{p+1}} \,)$

Fluid/mean-field approximation

- numeric vector representation can still result in a large number of states so use a fluid approximation [Hillston, 2005]
- treat subpopulation counts as real rather than integral and express change over time as ordinary differential equations (ODEs) giving one equation for each sequential state: $n' \times m$
- seldom obtain ODEs with analytical solutions but numerical ODE solution is generally fast
- ODE behaviour can approximate CTMC behaviour well if sufficient numbers (together with some other conditions as shown by Kurtz)

Fluid/mean-field approximation



Languages for modelling populations

- extensions to PEPA: multiple states per entity
  - Grouped PEPA [Hayden, Stefanek and Bradley, 2012]
  - Fluid process algebra [Tschaikowski and Tribastone, 2014]
- biological: single state and count per species
  - Bio-PEPA [Ciocchetta and Hillston, 2009]
  - Bio-PEPA with compartments [Ciocchetta and Guerriero, 2009]
- epidemiological: single state and count per subpopulation
  - variant of Bio-PEPA with locations [Ciocchetta and Hillston, 2010]

A stochastic population process algebra

- stochastic and deterministic semantics
- aim to be general but elementary
- each entity has a single state and a count
- is there a suitable equivalence?
- compression bisimulation [Galpin and Hillston, 2011]
- start more concretely and then consider more generality
- syntax from epidemiological modelling but different semantics

A stochastic population process algebra

- subpopulation description

$$C \stackrel{def}{=} (\beta_1, (\kappa_1, \lambda_1)) \odot C \;+\; \ldots \;+\; (\beta_{m_C}, (\kappa_{m_C}, \lambda_{m_C})) \odot C$$

  - actions: β_i are distinct
  - in and out stoichiometries: $\kappa_i, \lambda_i \in \mathbb{N}$

- composition of subpopulations

$$P \stackrel{def}{=} C_1(n_{1,0}) \underset{*}{\bowtie} \ldots \underset{*}{\bowtie} C_p(n_{p,0})$$

  - subpopulations: C_j are distinct,
  - initial quantities: $n_{j,0} \in \mathbb{N}$

- minimum and maximum size: M_C and N_C for each C
- range of a subpopulation is $N_C - M_C + 1$
- use $C^{(n)}$ to distinguish subpopulations with different ranges
- $P^{(n)}$ defines a composition whose minimum range is n

Operational semantics

$$\frac{}{C(n) \xrightarrow{\alpha,\{(C,n)\}}_c C(n - \kappa_k + \lambda_k)}$$

$$C \stackrel{def}{=} \sum_{k=1}^{n_C} (\beta_k, (\kappa_k, \lambda_k)) \odot C$$

$$\alpha \in \{\beta_1, \ldots, \beta_{n_C}\}$$

$$\kappa_k \leq n \leq N_C - \lambda_k$$

Operational semantics (continued)

$$\frac{P \xrightarrow{\alpha, W}_c P'}{P \underset{*}{\bowtie} Q \xrightarrow{\alpha, W}_c P' \underset{*}{\bowtie} Q} \qquad Q \not\xrightarrow{\alpha, W'}_c$$

$$\frac{Q \xrightarrow{\alpha, W}_c Q'}{P \underset{*}{\bowtie} Q \xrightarrow{\alpha, W}_c P' \underset{*}{\bowtie} Q} \qquad P \not\xrightarrow{\alpha, W'}_c$$

$$\frac{P \xrightarrow{\alpha, W_1}_c P' \qquad Q \xrightarrow{\alpha, W_2}_c Q'}{P \underset{*}{\bowtie} Q \xrightarrow{\alpha, W_1 \cup W_2}_c P' \underset{*}{\bowtie} Q'}$$

Operational semantics (continued)

$$\frac{P \xrightarrow{\alpha,W}_c P'}{P \xrightarrow{\alpha,f_\alpha(W)}_s P'}$$

- $f_\alpha : (\mathcal{C} \to \mathbb{N}) \to \mathbb{R}_{\geq 0}$ where $\mathcal{C}$ is the set of subpopulations
- f_α may make reference to M_C and N_C
- Markov chain semantics are given by $\xrightarrow{\alpha,r}_s$
- ODE semantics can be derived from $C_1(n_{1,0}) \bowtie_* \ldots \bowtie_* C_p(n_{p,0})$
- hybrid semantics by mapping to stochastic HYPE [Galpin 2014]
  - dynamic switching between stochastic and deterministic semantics for each action depending on subpopulation size or rate

Example

$$A \stackrel{def}{=} (\alpha_1, (1,0)) \odot A \; + \; (\alpha_2, (0,1)) \odot A \; + \; (\alpha_3, (2,0)) \odot A$$

$$B \stackrel{def}{=} (\alpha_3, (0,1)) \odot B$$

$$C \stackrel{def}{=} (\alpha_1, (0,1)) \odot C \; + \; (\alpha_2, (1,0)) \odot C$$

Example

$$A \stackrel{def}{=} (\alpha_1,(1,0)) \odot A \; + \; (\alpha_2,(0,1)) \odot A \; + \; (\alpha_3,(2,0)) \odot A$$

$$B \stackrel{def}{=} (\alpha_3,(0,1)) \odot B$$

$$C \stackrel{def}{=} (\alpha_1,(0,1)) \odot C \; + \; (\alpha_2,(1,0)) \odot C$$

Example

$$A \stackrel{def}{=} (\alpha_1,(1,0)) \odot A \;+\; (\alpha_2,(0,1)) \odot A \;+\; (\alpha_3,(2,0)) \odot A$$

$$B \stackrel{def}{=} (\alpha_3,(0,1)) \odot B$$

$$C \stackrel{def}{=} (\alpha_1,(0,1)) \odot C \;+\; (\alpha_2,(1,0)) \odot C$$

Example

$$
\begin{aligned}
A &\stackrel{def}{=} (\alpha_1,(1,0)) \odot A \;+\; (\alpha_2,(0,1)) \odot A \;+\; (\alpha_3,(2,0)) \odot A \\
B &\stackrel{def}{=} (\alpha_3,(0,1)) \odot B \\
C &\stackrel{def}{=} (\alpha_1,(0,1)) \odot C \;+\; (\alpha_2,(1,0)) \odot C
\end{aligned}
$$

Example

$$A \stackrel{def}{=} (\alpha_1,(1,0)) \odot A \;+\; (\alpha_2,(0,1)) \odot A \;+\; (\alpha_3,(2,0)) \odot A$$

$$B \stackrel{def}{=} (\alpha_3,(0,1)) \odot B$$

$$C \stackrel{def}{=} (\alpha_1,(0,1)) \odot C \;+\; (\alpha_2,(1,0)) \odot C$$

- consider $A(5) \underset{*}{\bowtie} B(0) \underset{*}{\bowtie} C(0)$ and $A(7) \underset{*}{\bowtie} B(0) \underset{*}{\bowtie} C(0)$
- express as labelled transition systems in numerical vector representation (n_A, n_B, n_C)

Example (continued)

$$
\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}
$$

Example (continued)

$$\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3\downarrow & & \alpha_3\downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccccc}
(7,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (6,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (5,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,5) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,6) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,7) \\
\alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & & & \\
(5,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,5) & & & & \\
\alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & \alpha_3\downarrow & & & & & & & & \\
(3,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,2,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,2,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,3) & & & & & & & & \\
\alpha_3\downarrow & & \alpha_3\downarrow & & & & & & & & & & & & \\
(1,3,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,3,1) & & & & & & & & & & & &
\end{array}$$

Example (continued)

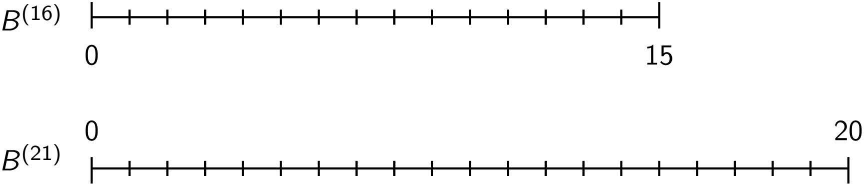
$$
\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}
$$

what is the equivalence that will identify these two models?

$$
\begin{array}{ccccccccccccccc}
(7,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (6,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (5,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,5) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,6) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,7) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(5,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,5) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(3,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,2,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,2,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,3) & & & & & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & & & & & \\
(1,3,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,3,1) & & & & & & & & & & & &
\end{array}
$$

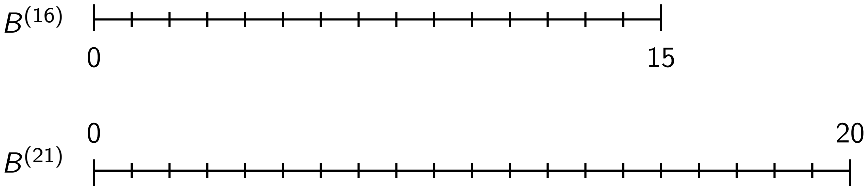
Single subpopulation

- $B \overset{def}{=} (\alpha, (3,0)) \odot B \; + \; (\beta, (0,4)) \odot B \; + \; (\gamma, (0,1)) \odot B$



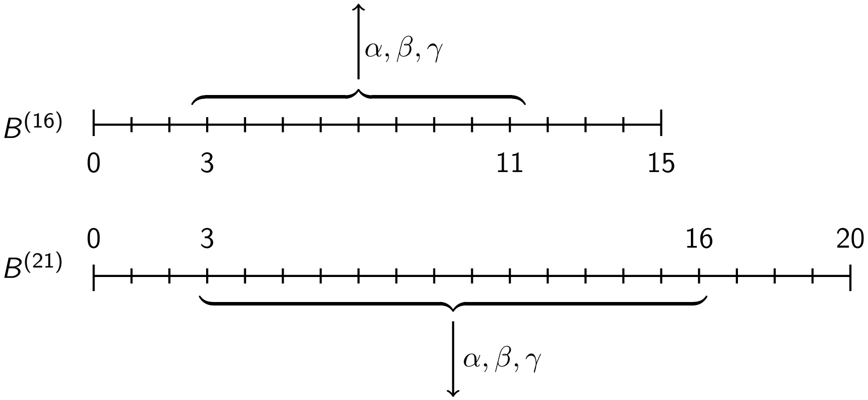
Single subpopulation

- $B \stackrel{def}{=} (\alpha, (3, 0)) \odot B \;+\; (\beta, (0, 4)) \odot B \;+\; (\gamma, (0, 1)) \odot B$



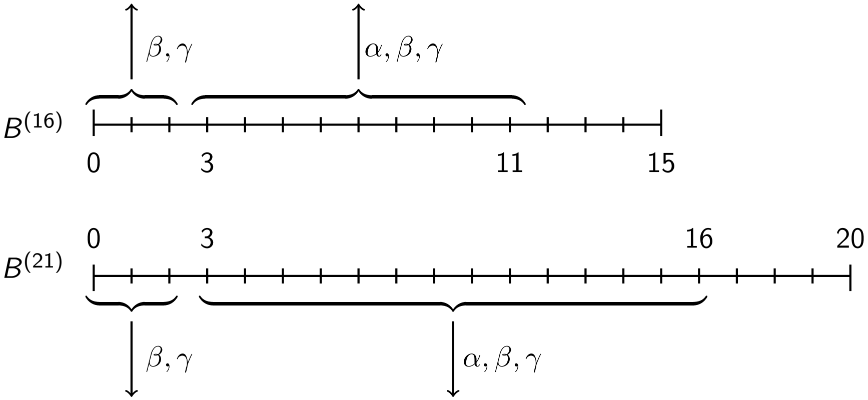
Single subpopulation

- $B \stackrel{def}{=} (\alpha, (3,0)) \odot B + (\beta, (0,4)) \odot B + (\gamma, (0,1)) \odot B$



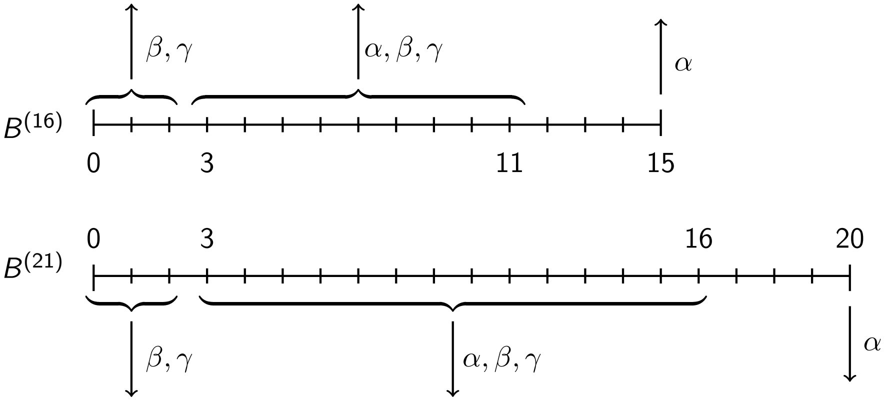
Single subpopulation

- $B \stackrel{def}{=} (\alpha, (3,0)) \odot B \; + \; (\beta, (0,4)) \odot B \; + \; (\gamma, (0,1)) \odot B$



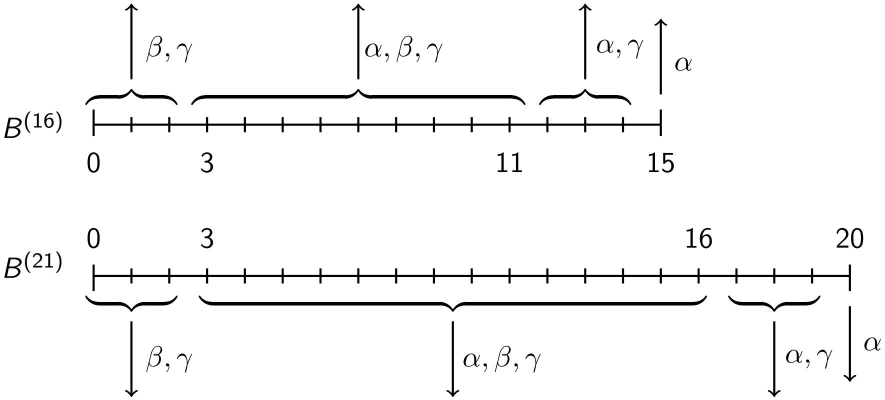
Single subpopulation

- $B \stackrel{def}{=} (\alpha, (3,0)) \odot B \; + \; (\beta, (0,4)) \odot B \; + \; (\gamma, (0,1)) \odot B$



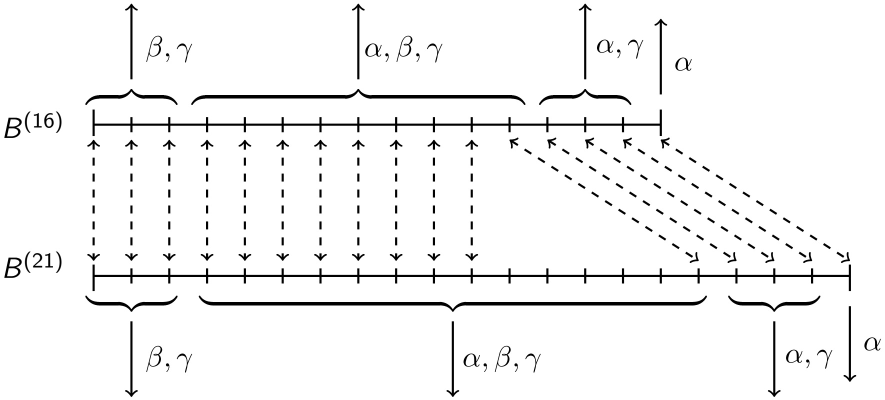
Single subpopulation

- $B \stackrel{def}{=} (\alpha, (3,0)) \odot B + (\beta, (0,4)) \odot B + (\gamma, (0,1)) \odot B$



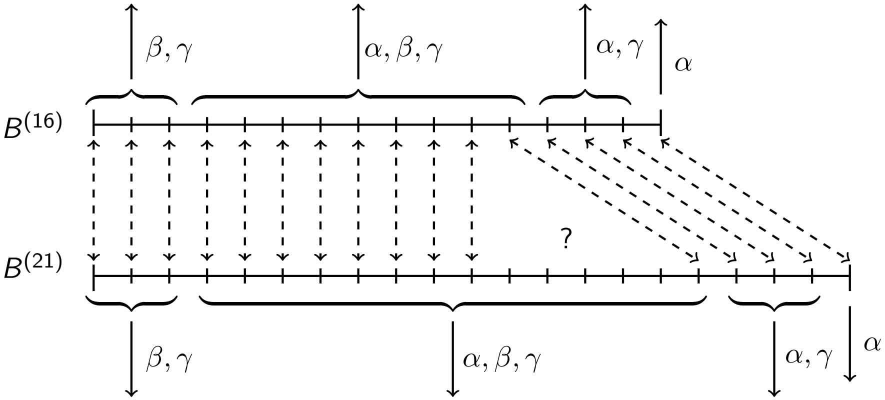
Single subpopulation

- $B \stackrel{def}{=} (\alpha,(3,0)) \odot B + (\beta,(0,4)) \odot B + (\gamma,(0,1)) \odot B$



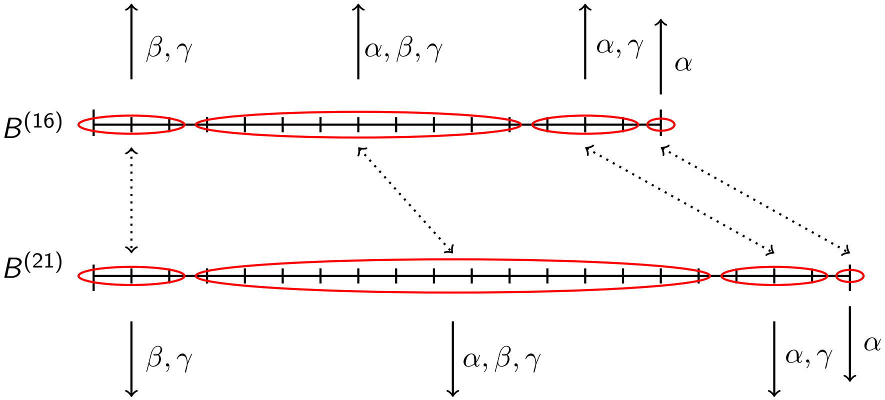
Single subpopulation

- $B \stackrel{def}{=} (\alpha,(3,0)) \odot B + (\beta,(0,4)) \odot B + (\gamma,(0,1)) \odot B$



Single subpopulation

- $B \stackrel{def}{=} (\alpha,(3,0)) \odot B \; + \; (\beta,(0,4)) \odot B \; + \; (\gamma,(0,1)) \odot B$



Compression bisimilarity

- $(P, Q) \in \mathcal{H}$ if they have same actions,
- define labelled transition system over equivalence classes of $\mathcal{H}$

$$[P] \xhookrightarrow{\alpha} [Q] \text{ if } P \xrightarrow{(\alpha, v)}_c Q$$

- compression bisimilarity, $P \doteq Q$ if $[P] \sim [Q]$, namely whenever
  1. $[P] \xhookrightarrow{\alpha} [P']$, then $[Q] \xhookrightarrow{\alpha} [Q']$ and $[P'] \sim [Q']$
  2. $[Q] \xhookrightarrow{\alpha} [Q']$, then $[P] \xhookrightarrow{\alpha} [P']$ and $[P'] \sim [Q']$
- results are given in terms of ranges

Results

- to show the full behaviour of a system $P^{(n)}$, n must be greater than the sum of
  - the maximum out-stoichiometry,
  - the maximum in-stoichiometry, and
  - the maximum in- or out-stoichiometry
- $C^{(n)} \doteq C^{(m)}$ if n and m are large enough
- $P^{(n)} \doteq P^{(m)}$ if n and m are large enough together with a technical condition required for stoichiometries larger than 1
- $\doteq$ is a congruence for $\bowtie_{*}$ if technical condition holds

Example (revisited)

$$
\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}
$$

Example (revisited)

$$\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
(6,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (5,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,5) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,6) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(4,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,4) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(2,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,2,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,2) & & & & & & & & \\
\alpha_3 \downarrow & & & & & & & & & & & & \\
(0,3,0) & & & & & & & & & & & &
\end{array}$$

Example (revisited)

$$
\begin{array}{ccccccccccc}
(5,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,5) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(3,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,3) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(1,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,1) & & & & & & & &
\end{array}
$$

these are not compression bisimilar

$$
\begin{array}{ccccccccccccc}
(6,0,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (5,0,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (4,0,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,0,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,0,4) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,0,5) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,0,6) \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & \\
(4,1,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (3,1,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (2,1,2) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,1,3) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,1,4) & & & & \\
\alpha_3 \downarrow & & \alpha_3 \downarrow & & \alpha_3 \downarrow & & & & & & & & \\
(2,2,0) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (1,2,1) & \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} & (0,2,2) & & & & & & & & \\
\alpha_3 \downarrow & & & & & & & & & & & & \\
(0,3,0) & & & & & & & & & & & &
\end{array}
$$

Open problems

- hypothesis: if T is the lcm for all stoichiometric coefficients, $n = m + cT$ for $c \in \mathbb{N}$ and n, m large enough, then $P^n \doteq P^m$

  can this be proved?

Open problems

- hypothesis: if T is the lcm for all stoichiometric coefficients, $n = m + cT$ for $c \in \mathbb{N}$ and n, m large enough, then $P^n \doteq P^m$

  can this be proved?

- can compression bisimulation be extended to an (approximate) quantitative equivalence?

Open problems

- hypothesis: if T is the lcm for all stoichiometric coefficients, $n = m + cT$ for $c \in \mathbb{N}$ and n, m large enough, then $P^n \doteq P^m$

  can this be proved?

- can compression bisimulation be extended to an (approximate) quantitative equivalence?

- are there other operators of interest?
  - can two subpopulations, C and D, be combined?
  - define a new operator $C \boxplus D$
  - must the actions of C and D be disjoint?
  - can a single subpopulation have repeated actions?

Open problems (continued)

- how can the notion of a stochastic population process algebra be made more general?
- what are the important aspects?
- can these be expressed by parameterising functions?
- choice of functions instantiates population process algebra
- provide meta-results with respect to these functions
- not as general as a SOS format

More generally

$$\frac{}{C(n) \xrightarrow{\alpha, \nu_\alpha^C(n)}_c C(\mu_\alpha^C(n))}$$

$$C \stackrel{def}{=} \sum_{k=1}^{n_C} \beta_k \odot C$$

$$\alpha \in \{\beta_1, \ldots, \beta_{n_C}\}$$

$\mu_\alpha^C(n)$ and $\nu_\alpha^C(n)$ are defined

- stoichiometric information and conditions no longer appear in the prefix but are embedded in the definition of the function μ_α^C
- only local information about C can be used in μ_α^C and ν_α^C

More generally (continued)

$$\frac{P \xrightarrow{\alpha,S}_c P'}{P \bowtie_* Q \xrightarrow{\alpha,S}_c P' \bowtie_* Q} \qquad Q \not\xrightarrow{\alpha,S'}_c$$

$$\frac{Q \xrightarrow{\alpha,S}_c Q'}{P \bowtie_* Q \xrightarrow{\alpha,S}_c P' \bowtie_* Q} \qquad P \not\xrightarrow{\alpha,S'}_c$$

$$\frac{P \xrightarrow{\alpha,S_1}_c P' \quad Q \xrightarrow{\alpha,S_2}_c Q'}{P \bowtie_* Q \xrightarrow{\alpha,\rho_\alpha(S_1,S_2)}_c P' \bowtie_* Q'} \qquad \text{if } \rho_\alpha(S_1,S_2) \text{ is defined}$$

More generally (continued)

$$\frac{P \xrightarrow{\alpha,S}_c P'}{P \xrightarrow{\alpha,f_\alpha(S)}_s P'}$$

- $f_\alpha : \mathcal{S} \to \mathbb{R}_{\geq 0}$
- Markov chain semantics are given by $\xrightarrow{\alpha,r}_s$
- ODEs can be derived from $C_1(n_{1,0}) \bowtie_{*} \ldots \bowtie_{*} C_p(n_{p,0})$
- unspecified functions: ν_α^C, μ_α^C, ρ_α, f_α
- what are sensible choices in the context of population modelling?

Open problems (continued)

- how can modelling of space in the context of smart transport and smart grids be combined with population modelling?

Open problems (continued)

- how can modelling of space in the context of smart transport and smart grids be combined with population modelling?

Thank you