Characterising State Spaces of Concurrent Systems

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Work started with Philippe Darondeau
and continued with Raymond Devillers

Open Problems in Concurrency Theory
Bertinoro, June 18, 2014

System analysis vs. system synthesis

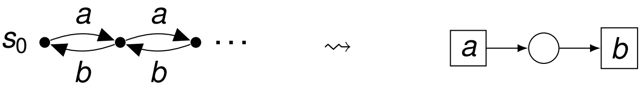
- **Analysis**
  Given: a system (program, algorithm, expression, Petri net)
  Objective: deduce behavioural properties

  State space exploration / representation / explosion
- **Synthesis**
  Given: a specification describing desired behaviour
  Objective: derive a generating / implementing system

  Correctness by design

Synthesis of Petri nets

- Input A labelled transition system $(S, \rightarrow, T, s_0)$ with states S (initially s_0), labels T, arcs $\rightarrow \subseteq (S \times T \times S)$
- Output A marked Petri net with transitions T and isomorphic state space



Region theorems for an lts $TS = (S, \rightarrow, T, s_0)$

- $(\mathbb{R}, \mathbb{B}, \mathbb{F}) \in (S \rightarrow \mathbb{N}, T \rightarrow \mathbb{N}, T \rightarrow \mathbb{N})$ region of TS if
  $s \xrightarrow{t} s' \quad \Rightarrow \quad \mathbb{R}(s) \geq \mathbb{B}(t)$ and $\mathbb{R}(s') = \mathbb{R}(s) - \mathbb{B}(t) + \mathbb{F}(t)$

A region 'behaves like a Petri net place' but is defined on TS

- TS satisfies ESSP (event/state separation property) if
  $\neg(s \xrightarrow{t}) \quad \Rightarrow \quad \exists$ region $(\mathbb{R}, \mathbb{B}, \mathbb{F})$ with $\mathbb{R}(s) < \mathbb{B}(t)$
- ... and SSP (state separation property) if
  $s \neq s' \quad \Rightarrow \quad \exists$ region $(\mathbb{R}, \mathbb{B}, \mathbb{F})$ with $\mathbb{R}(s) \neq \mathbb{R}(s')$

Theorems (for finite lts):

ESSP $\Rightarrow$ $\exists$ a language-equivalent Petri net

ESSP$\wedge$SSP $\Rightarrow$ $\exists$ a Petri net with isomorphic reachability graph

Ehrenfeucht, Rozenberg et al.
Upcoming book by Badouel, Bernardinello, Darondeau

Checking the region properties, and open problems

- As far as I am aware, this theory has not yet been fully extended to infinite transition systems (but: Darondeau)
- For finite-state systems, the basic algorithm is polynomial
- BUT in the size of the lts!
- AND with exponents 7 or 8!
- The region theorems are pretty unwieldy
- Apparently, there is even no characterisation yet of the case that a finite straight lts (a word) satisfies ESSP
- If an lts is Petri net realisable there are usually many incomparable minimal solutions

Our approach Identify classes of lts for which structurally pleasant solutions can be shown to exist

A live and bounded marked graph

M_0

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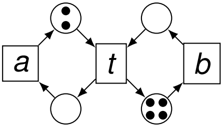
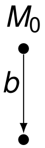


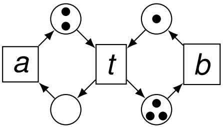
A marked graph Petri net

and its initial marking M_0

marked graph:
a Petri net with plain arcs and
$|^\bullet p| = 1 = |p^\bullet|$ for all places p
where $^\bullet p$ = input transitions of p
and $p^\bullet$ = output transitions of p

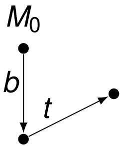
A live and bounded marked graph

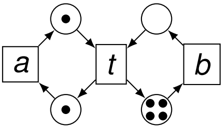




after executing b

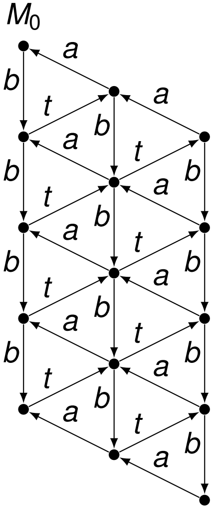
A live and bounded marked graph

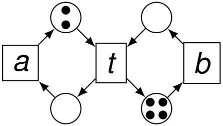




after executing *bt*

A live and bounded marked graph



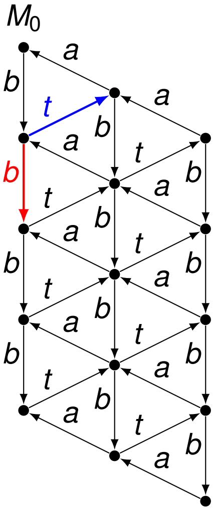


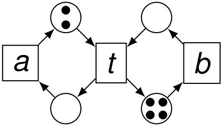
A marked graph Petri net

and its reachability graph..

..which has several nice properties:

It is **deterministic**

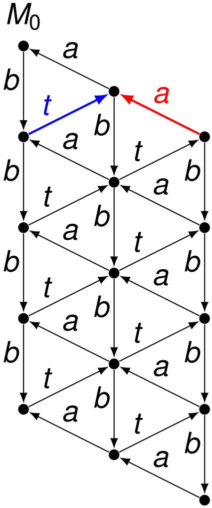


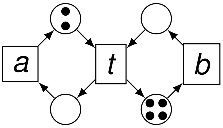


Determinism If a state enables b and t, leading to different states, then $b \neq t$

.. true because the reachability graph comes from a Petri net

... and **backward deterministic**

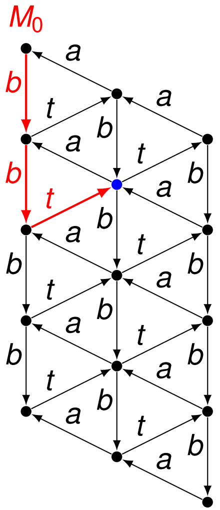


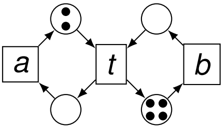


Backward determinism If *a* and *t* lead to a state from different states, then $a \neq t$

.. true because the reachability graph comes from a Petri net

It is **totally reachable**

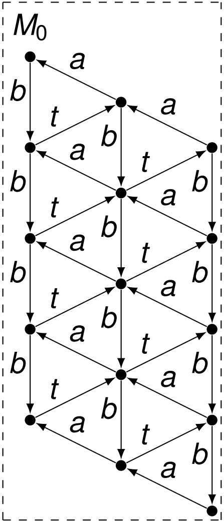


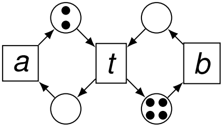


Total reachability Every state is reachable from the initial state M_0

.. true by the definition of reachability graph

It is **finite**

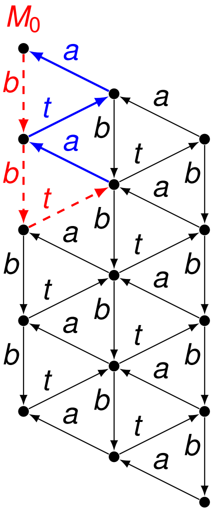


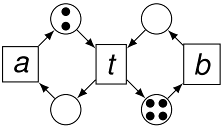


Finiteness

..due to the boundedness of the net

It is **reversible**

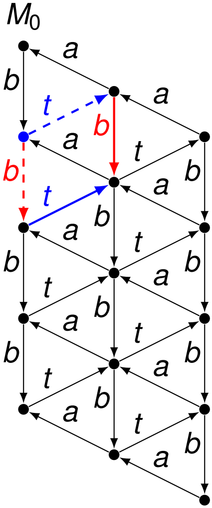


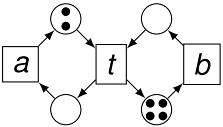


Reversibility The initial state is reachable from every reachable state

.. true (for marked graphs) by liveness and boundedness

It is **persistent**



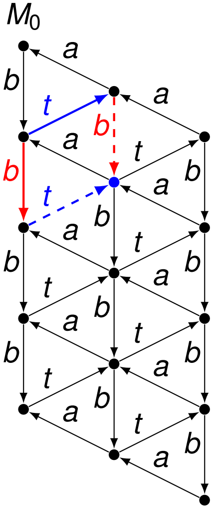


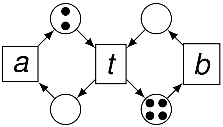
Persistency If a state enables b and t for $b \neq t$, then it also enables bt and tb

.. true by the marked graph property

also called *strong confluence*

It is **backward persistent**

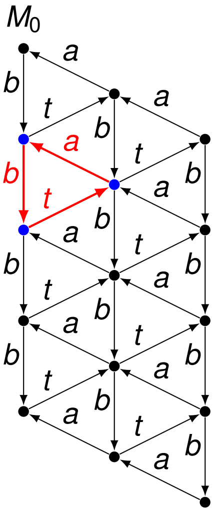


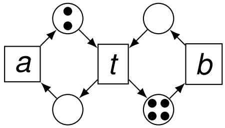


Backward persistency
If a state backward enables b and t for $b \neq t$, from two reachable states, then it also backward enables bt and tb

.. true by the marked graph property

It satisfies the **P1 property**





The Parikh 1 property
In a small cycle, every firable transition occurs exactly once

.. true by the marked graph property

Note: $M_0 \xrightarrow{bbttaa} M_0$ is not small

small means:
nonempty and Parikh-minimal

State spaces of live and bounded marked graphs

Theorem The following are equivalent:

A *TS* is isomorphic to the reachability graph of a live and bounded marked graph

B *TS* is

- deterministic and backward deterministic
- totally reachable
- finite
- reversible
- persistent
- backward persistent
- and satisfies the P1 property of small cycles

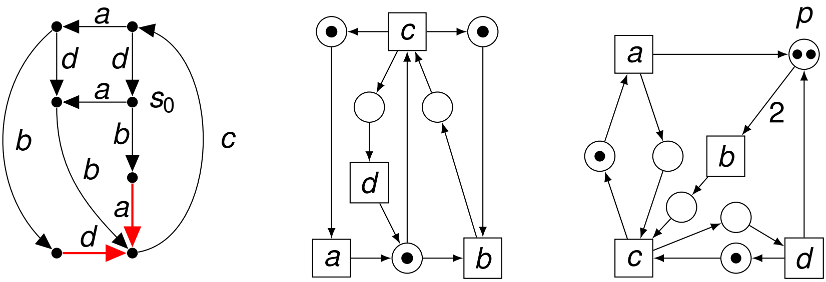
The proof of **A**⇒**B** is in Commoner, Genrich et al. (1968–...)

The proof of **B**⇒**A** is in LATA' 2014 (constructing regions)

Moreover: ∃ a unique minimal marked graph realising *TS*

Necessity of backward persistency

The lts shown below satisfies all properties of **B** except backward persistency

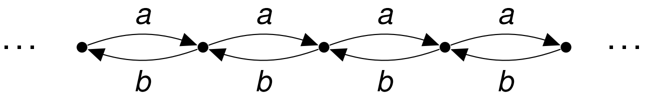


There is no marked graph solution

There are two different minimal non-marked graph solutions

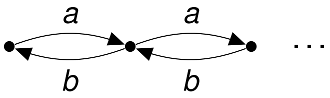
(Non-) solvable infinite lts

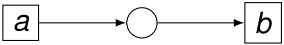
- The following infinite lts is not Petri net solvalbe:



Uniform 2-way infinite chains such as $\ldots aaaa \ldots$ or $\ldots bbbb \ldots$ cannot be part of a Petri net state space

- The following infinite lts is Petri net solvalbe:





Non-uniform 2-way infinite chains $\ldots bbaa \ldots$ are acceptable

State spaces of live, unbounded marked graphs

Theorem The following are equivalent:

A *TS* is isomorphic to the reachability graph of a live, unbounded marked graph

B *TS* is

- deterministic and backward deterministic
- totally reachable
- infinite, but has no uniform 2-way infinite chains $\ldots\alpha\alpha\alpha\alpha\ldots$
- reversible
- persistent
- backward persistent
- and satisfies the P1 property of small cycles

The proof of (**A**⇒**B**) is ‘common knowledge’

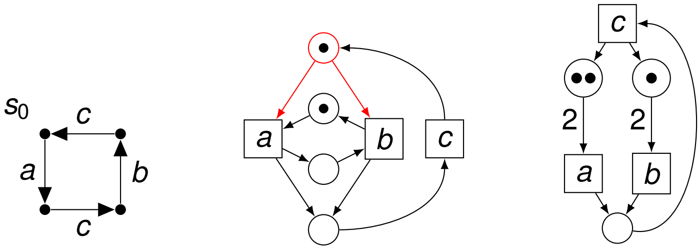
The proof of (**B**⇒**A**) is in a submitted paper (June 2014)

Moreover: $\exists$ a unique minimal marked graph realising *TS*

Necessity of the P1 property

The lts shown below satisfies all properties of **B** except P1

By definition, it satisfies PΥ with $\Upsilon = (\#a, \#b, \#c) = (1, 1, 2)$



There is no marked graph solution

There are two different minimal non-marked graph solutions

The middle solution has a 'fake' (but non-redundant) choice

The r.h.s. solution is 'nicer' in the sense that it satisfies $|p^\bullet| \leq 1$

State spaces of reversible, bounded, ON Petri nets

ON (output-nonbranching): $|p^\bullet| \leq 1$ for all places p
(weakens the defining marked graph properties)

Theorem The following are equivalent:

A *TS* is isomorphic to the reachability graph of a reversible, bounded ON net

B *TS* is

- deterministic and totally reachable
- finite, reversible and persistent
- and satisfies the PΥ property of small cycles, with a constant Υ
- such that Υ enjoys $\gcd_{t \in T}\{\Upsilon(t)\} = 1$
- and for every $x \in T$ and maximal non-x-enabling state s the system

$$\forall r \in NUI(x)\colon 0 < \sum_{1 \leq j \leq |T|} k_j \cdot (\Upsilon(t_j) \cdot (1 + \Delta_{r,s}(x)) - \Upsilon(x) \cdot \Delta_{r,s}(t_j))$$

has a nonnegative integer solution $k_1, \ldots, k_{|T|}$

Υ: a Parikh vector (not necessarily 1, but the same for all small cycles)
$NUI(x)$: non-x-enabling states with a unique incoming arrow labelled x
$\Delta_{r,s}$: Parikh-distance between r and s (well-defined by some properties in **B**)
Proof: Using region theory again; see Petri Nets 2014 (Tunis, next week)
The inequalities in **B** only refer to proper (and 'small') subsets of states

Concluding remarks, and open problems

- The last result characterises finite, reversible, arbitrarily Petri net distributable (in the sense of Hopkins, Badouel et al.) lts
- Some lts are distributable but not arbitrarily so, and existing results would need to be extended
- Results tend to come with fast, dedicated synthesis algorithms
- ... whose complexity can not necessarily be analysed easily because of interdependencies of the sizes of special lts subsets
- Bounded non-labelled Petri nets also seem to give rise to a hierarchy inside regular languages that has, to my knowledge, not yet been deeply studied

In Petri net theory, several key (decidability) problems are still open
My favourite: the existence of a home state
Another favourite: language-equivalence under restrictions

The Nielsen, Thiagarajan conjecture still seems to be unsolved, too ...
Their conjecture has a flavour similar to the characterisation results mentioned in this talk, except that lts are replaced by event structures and a different class of Petri nets is concerned